

Matrix Ernst potentials for EMDA with multiple vector fields

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Abstract

We show that the Einstein–Maxwell–Dilaton–Axion system with multiple vector fields (bosonic sector of the $D = 4, N = 4$ supergravity) restricted to spacetimes possessing a non-null Killing vector field admits a concise representation in terms of the Ernst-type matrix valued potentials. A constructive derivation of the SWIP solutions is given and a colliding waves counterpart of the DARN-NUT solution is obtained. $SU(m, m)$ chiral representation of the two-dimensionally reduced system is derived and the corresponding Kramer–Neugebauer-type map is presented.

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Recently a variety of black hole solutions was found in the four-dimensional extended supergravities [1] using either *ad hoc* ansatze or employing classical dualities. In the most extensively studied $N = 4$ theory it was shown that the corresponding three-dimensional reduction (with a non-null spacetime Killing symmetry assumed) may be concisely formulated in terms of generalized Ernst potentials [2]. This suggests an alternative interpretation of the classical U -duality as the ‘Ehlers’ symmetry and opens a way to apply powerful general relativity techniques to construct exact classical solutions. For the Einstein–Maxwell–Dilaton–Axion (EMDA) theory with one vector field a particularly simple matrix Ernst potential was found in terms of 2×2 symmetric complex matrices [3]. This representation, however, is due to existence of an exceptional local isomorphism $SO(2, 3) \sim Sp(4, R)$, relevant to the one-vector EMDA U -duality $SO(2, 3)$ [4], which is not extendible to the realistic case of multiple vector fields. Here we show that in the case of two vector fields ($p = 2$) another exceptional local isomorphism $SO(2, 4) \sim SU(2, 2)$ gives rise to an even more economical representation of the 8-dimensional TS in terms of the 2×2 complex *non-symmetric* matrices (reducing to symmetric ones for $p = 1$). For arbitrary p a matrix Ernst potential can be constructed using the Clifford algebras corresponding to the compact subgroup $SO(p + 1)$ of the three-dimensional T -duality group $SO(1, p + 1)$. This leads to the pseudounitary embedding of the U -duality group $SO(2, 2 + p)$ into $SU(m, m)$ where $m = 2^k$, $k = [(p + 1)/2]$. In terms of the matrix Ernst potential U -duality looks like a matrix-valued ‘Ehlers’ $SL(2, R)$ symmetry [5]. Further two-dimensional reduction of the theory (with the rank-two Abelian spacetime

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isometry group assumed) leads to the $SU(m, m)$ chiral representation in the σ -model variables, or to its ‘Matzner–Misner’ counterpart obtainable via the Kramer–Neugebauer-type map.

We start with the four-dimensional action

$$S = \int \left\{ -R + 2 \left| \partial z (z - \bar{z})^{-1} \right|^2 + \left(i z \mathcal{F}_{\mu\nu}^n \mathcal{F}^{n\mu\nu} + c.c \right) \right\} \sqrt{-g} d^4 x, \quad (1)$$

where $\mathcal{F}^n = (F^n + i\tilde{F}^n)/2$, $\tilde{F}^{n\mu\nu} = \frac{1}{2} E^{\mu\nu\lambda\tau} F_{\lambda\tau}^n$, $n = 1, \dots, p$ (summation over repeated n is understood elsewhere), $z = \kappa + i e^{-2\phi}$, and the metric signature is $+- --$. For $p = 6$ this action describes the bosonic sector of $N = 4, D = 4$ supergravity. It is invariant under the $SO(p)$ rotation of the vector fields, which is an analogue of the T -duality of dimensionally reduced theories [6]. The equations of motion and Bianchi identities (but not the action) are also invariant under the S -duality transformations

$$z \rightarrow \frac{az + b}{cz + d}, \quad ad - bc = 1, \\ F^n \rightarrow (c\kappa + d)F^n + ce^{-2\phi}\tilde{F}^n. \quad (2)$$

Consider three-dimensional reduction of the theory assuming either timelike ($\lambda = 1$), or spacelike ($\lambda = -1$) (in an essential region of spacetime) Killing symmetry. Then the four-dimensional line element may be written as

$$ds^2 = \lambda f (dy - \omega_i dx^i)^2 - \frac{\lambda}{f} h_{ij} dx^i dx^j, \quad (3)$$

where the three-space metric h_{ij} ($i, j = 1, 2, 3$), the one-form ω_i and the conformal factor f depend on the three-space coordinates x^i only. It is assumed that $y = t$, h_{ij} is spacelike for $\lambda = 1$, and h_{ij} is of the signature $+- --$ for $\lambda = -1$.

One can express vector fields through the quantities v^n, u^n which have the meaning of the electric and magnetic scalar potentials for $\lambda = 1$:

$$F_{iy}^n = \frac{1}{\sqrt{2}} \partial_i v^n, \quad (4)$$

$$2\text{Im} \left(z \mathcal{F}^{nij} \right) = \frac{f}{\sqrt{2}h} \epsilon^{ijk} \partial_k u^n, \quad h \equiv \det h_{ij}. \quad (5)$$

In three dimensions the ‘ T -duality’ group is enlarged to $SO(1, p+1)$, while the S -duality becomes the symmetry of the action. Moreover, both these groups are unified within a larger ‘ U -duality’ group $SO(2, p+2)$ [3, 7, 8]. This can be easily shown by constructing the Kähler metric of the target manifold of the resulting σ -model. To find such a representation one has to introduce a twist potential χ via

$$d\chi = u^n dv^n - v^n du^n - \lambda f^2 * d\omega, \quad (6)$$

and to derive equations for χ, u^n in addition to those for f, κ, ϕ, v^n . The full set of equations will be that of the three-dimensional gravity coupled non-linear σ -model possessing the $4 + 2p$ dimensional target space $SO(2, 2+p)/(SO(2) \times SO(p, 2))$ for $\lambda = 1$, respectively $SO(2, 2+p)/(SO(2) \times SO(p+2))$ for $\lambda = -1$. In the latter case the corresponding matrices are symmetric, what is a desirable property for an application of the inverse scattering transform technique. Since the transition from $\lambda = 1$ to $\lambda = -1$ in (3) is merely an analytic continuation, symmetric matrices may be used in the $\lambda = 1$ case as well (a realization of

the non-compact coset by symmetric matrices may be achieved via the left multiplication by some constant matrix).

The target manifold can be parametrized by complex coordinates z^α , $\alpha = 0, 1, \dots, p+1$ which have the following meaning. The components $\alpha = n = 1, \dots, p$ are complex potentials for vector fields

$$z^n = u^n - zv^n \equiv \Phi^n, \quad n = 1, \dots, p, \quad (7)$$

while the $\alpha = 0, p+1$ components are linear combinations of the complex axidilaton and the Ernst potential $E = i\lambda f - \chi + v^n \Phi^n$:

$$z^0 = (E - z)/2, \quad z^{p+1} = (E + z)/2. \quad (8)$$

The TS metric is generated by the Kähler potential [2]

$$G_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K(z^\alpha, \bar{z}^\beta), \quad (9)$$

$$K = -\ln V, \quad V = \lambda \eta_{\alpha\beta} \text{Im} z^\alpha \text{Im} z^\beta = f e^{-2\phi}, \quad (10)$$

where the T -duality $SO(1, p+1)$ metric is introduced $\eta_{\alpha\beta} = \text{diag}(-1, 1, \dots, 1)$, $\alpha, \beta = 0, 1, \dots, p+1$.

For $p = 1$ the matrix Ernst potential incorporating linearly all Kähler variables reads [3, 7]

$$\mathcal{E} = \begin{pmatrix} E & \Phi \\ \Phi & -z \end{pmatrix}. \quad (11)$$

This is a symmetric complex matrix which splits into hermitean and antihermitean parts

$$\mathcal{E} = \mathcal{Q} + i\mathcal{P}, \quad \mathcal{P}^\dagger = \mathcal{P}, \quad \mathcal{Q}^\dagger = \mathcal{Q}, \quad (12)$$

with *real* \mathcal{Q} , \mathcal{P} and generates a symmetric $Sp(4, R)$ matrix

$$\mathcal{M} = \begin{pmatrix} \mathcal{P}^{-1} & \mathcal{P}^{-1}\mathcal{Q} \\ \mathcal{Q}\mathcal{P}^{-1} & \mathcal{P} + \mathcal{Q}\mathcal{P}^{-1}\mathcal{Q} \end{pmatrix}, \quad (13)$$

satisfying

$$\mathcal{M}^\dagger J \mathcal{M} = J, \quad J = \begin{pmatrix} O & I_2 \\ -I_2 & O \end{pmatrix}. \quad (14)$$

It can be checked that the TS metric is

$$dl^2 = -2\text{Tr} \left\{ d\mathcal{E} (\mathcal{E}^\dagger - \mathcal{E})^{-1} d\mathcal{E}^\dagger (\mathcal{E}^\dagger - \mathcal{E})^{-1} \right\}. \quad (15)$$

Kähler potentials act as scalar sources in the three-dimensional Einstein's equations

$$\mathcal{R}_{ij} = -2\text{Tr} \left\{ (\mathcal{E}^\dagger - \mathcal{E})^{-1} (\partial_i \mathcal{E}) (\mathcal{E}^\dagger - \mathcal{E})^{-1} \partial_j \mathcal{E}^\dagger \right\}. \quad (16)$$

Alternatively, in terms of \mathcal{M} , the TS metric reads

$$dl^2 = -\frac{1}{4} \text{Tr} \{ d\mathcal{M} d\mathcal{M}^{-1} \}, \quad (17)$$

while the Einstein's equations for h_{ij} are

$$\mathcal{R}_{ij} = -\frac{1}{4} \text{Tr} \{ (\partial_i \mathcal{M}) \partial_j \mathcal{M}^{-1} \}. \quad (18)$$

Here we are looking for a generalization of this representation to higher p . It turns out that this can be achieved not in terms of higher rank symplectic groups, but rather in terms of pseudounitary imbeddings. Consider first the case $p = 2$. Then the global symmetry of the TS (U -duality) is the four-dimensional conformal group $SO(2, 4) \sim SU(2, 2)$. The latter group, realised by (4×4) complex matrices, can be conveniently presented using the Dirac basis $\sigma_{\mu\nu} = \rho_\mu \otimes \sigma_\nu$, where ρ_μ, σ_ν are two sets of Pauli matrices (with the unit matrix for $\mu, \nu = 0$) [9]. Any element $\mathcal{U} \in SU(2, 2)$ satisfies $\mathcal{U}^\dagger \sigma_{30} \mathcal{U} = \sigma_{30}$. To get contact with $p = 1$ one has to perform the unitary transformation

$$\mathcal{M} = V^\dagger \mathcal{U} V, \quad V = (\sigma_{00} - i\sigma_{10})/\sqrt{2}, \quad (19)$$

so that \mathcal{M} should obey (14) (in the context of unitary groups it is more natural to multiply J by i , *i.e.* to take $J = \sigma_{20}$). Then the expression (15) for the TS line element remains valid (up to a numerical factor) for the following $p = 2$ matrix Ernst potential:

$$\mathcal{E} = \begin{pmatrix} E & \Phi_1 - i\Phi_2 \\ \Phi_1 + i\Phi_2 & -z \end{pmatrix}. \quad (20)$$

With the same block parametrization (13) the formulas (17,18) also hold up to a normalization. Note that now hermitean \mathcal{P}, \mathcal{Q} are not real.

The essential feature of the matrix Ernst representation is that it provides the *matrix-valued* generalization of the Ehlers group of the vacuum general relativity [2]. This gives an alternative view on the U -duality in three-dimensional supergravities. For $p = 2$ the 15-parametric ‘Ehlers’ group consists of the four-parametric gauge,

$$\mathcal{E} \rightarrow \mathcal{E} + \mathcal{G}, \quad \mathcal{G} = \begin{pmatrix} g & m^1 - im^2 \\ m^1 + im^2 & b \end{pmatrix}, \quad (21)$$

(g, b are twist and axion shift parameters, m^n is a magnetic gauge), the four-parametric ‘proper Ehlers’ (including the ‘Ehlers’-like S -duality component),

$$\mathcal{E}^{-1} \rightarrow \mathcal{E}^{-1} + \mathcal{H}, \quad \mathcal{H} = \begin{pmatrix} c_E & h_m^1 - ih_m^2 \\ h_m^1 + ih_m^2 & c \end{pmatrix}, \quad (22)$$

and the seven-parametric ‘scale’ transformation:

$$\mathcal{E} \rightarrow \mathcal{S}^\dagger \mathcal{E} \mathcal{S}, \quad \mathcal{S} = \begin{pmatrix} e^{s+i\alpha} & h_e^1 - ih_e^2 \\ -e^1 + ie^2 & ae^{-i\alpha} \end{pmatrix}. \quad (23)$$

Note, that the Harrison transformations of this theory [2, 10] (parametrized by $h_e^n, h_m^n, n = 1, 2$) enter partly into the ‘Ehlers’ and partly into the ‘scale’ subgroups. In the latter the parameter α represents the $SO(p)$ ($p = 2$) rotations (the four-dimensional ‘ T -duality’).

To get the desired generalization to arbitrary p the following observation is appropriate. The structure of the matrix Ernst potential for $p = 2$ may be viewed as an expansion over the Clifford algebra corresponding to the $SO(p+1)$ subgroup of the three-dimensional T -duality group:

$$\{\gamma_a, \gamma_b\} = 2\delta_{ab}I_m, \quad (24)$$

where $a, b = 1, \dots, p+1$, $m = 2^k$, $k = [(p+1)/2]$. For $p = 2$ the Clifford algebra is realized by the Pauli matrices σ_a , and clearly

$$\mathcal{E} = z^0 I_2 + z^a \sigma_a. \quad (25)$$

For arbitrary p one has merely to replace σ_a by *hermitean* γ_a :

$$\mathcal{E} = z^0 I_k + z^a \gamma_a. \quad (26)$$

The dimensionality of this representation follows the usual rule valid for gamma-matrices in arbitrary dimensions: for even $p = 2k$ gamma-matrices are $2^k \times 2^k$, while for $p = 2k + 1$ the rank is the same as for $p = 2(k + 1)$. The only modification to be made in (15-18) is a numerical factor due to the trace of the unit matrix:

$$dl^2 = \frac{1}{m} \text{Tr} \{ d\mathcal{E} \mathcal{P}^{-1} d\mathcal{E}^\dagger \mathcal{P}^{-1} \} = -\frac{1}{2m} \text{Tr} \{ d\mathcal{M} d\mathcal{M}^{-1} \}, \quad (27)$$

$$\mathcal{R}_{ij} = \frac{1}{m} \text{Tr} \{ (\partial_{(i} \mathcal{E}) \mathcal{P}^{-1} (\partial_{j)} \mathcal{E}^\dagger) \mathcal{P}^{-1} \} = -\frac{1}{2m} \text{Tr} \{ (\partial_{(i} \mathcal{M}) \partial_{j)} \mathcal{M}^{-1} \}. \quad (28)$$

The corresponding expansions of \mathcal{Q}, \mathcal{P} are given by (26) again with the real and imaginary parts of z^α respectively. Matrices \mathcal{M} are hermitean by construction (for both $\lambda = \pm 1$) and belong to $SU(m, m)$. The complex matrices \mathcal{E} are ‘filled densely’ only for $p = 2$, in which case the number of complex potentials coincides with the number of matrix elements (four). For $p > 2$ one has $m^2 > p + 2$. This reflects the fact that the local isomorphism between $SO(2, p + 2)$ and non-compact unitary groups holds uniquely for $p = 2$, for higher p we deal only with *embeddings* into $SU(m, m)$.

Consider now the case $p = 2$ in more detail. The algebra $su(2, 2)$ is formed by the complex traceless 4×4 matrices X subject to the condition

$$X^\dagger \sigma_{20} + \sigma_{20} X^\dagger = 0. \quad (29)$$

It consists of 8 hermitean $\sigma_{10}, \sigma_{30}, \sigma_{11}, \sigma_{31}, \sigma_{12}, \sigma_{32}, \sigma_{13}, \sigma_{33}$, and 7 antihermitean $i(\sigma_{01}, \sigma_{02}, \sigma_{03}, \sigma_{21}, \sigma_{22}, \sigma_{23}, \sigma_{20})$ generators. Its subsequent decomposition will be performed in relation to the geodesic ansatz for the matrix \mathcal{M} :

$$\mathcal{M} = A e^{B\sigma}. \quad (30)$$

(More about geodesic technique with a detailed discussion of the $p = 1$ theory see in [9]). In (30) σ is a harmonic function on the three-space, normalized to zero in some (‘empty’) region of the spacetime (where $\mathcal{M} = A$), and B is an element of $su(2, 2)$. We will be interested in two types of solutions: stationary asymptotically flat (SAF) configurations (elliptic case, $\lambda = 1$), and colliding plane waves (CPW) (hyperbolic case, $\lambda = -1$). For SAF $A = \sigma_{03}$, while for CPW $A = -\sigma_{00}$ (this is equivalent to say that in the ‘empty’ region $f = 1, \chi = \phi = \kappa = v^n = u^n = 0$). In both cases it is convenient to use a representation with the hermitean $\mathcal{M} \in SU(2, 2)$, therefore the matrix B has to satisfy the following conditions

$$AB = B^\dagger A, \quad BK + KB = 0, \quad (31)$$

where $K = \sigma_{23}$ for SAF, and $K = -\sigma_{20}$ for CPW. In the SAF case the elements of $su(2, 2)$ satisfying these conditions form two sets of $SO(2, 2)$ Clifford algebras

$$\Gamma_\mu^1 = \{i\sigma_{21}, i\sigma_{22}, -\sigma_{10}, \sigma_{30}\}, \quad \Gamma_\mu^2 = \{i\sigma_{02}, -i\sigma_{01}, \sigma_{33}, \sigma_{13}\}, \quad (32)$$

obeying

$$\{\Gamma_\mu^1, \Gamma_\nu^1\} = \{\Gamma_\mu^2, \Gamma_\nu^2\} = 2\eta_{\mu\nu} I \quad (33)$$

with $\eta_{\mu\nu} = \text{diag}(-1, -1, 1, 1)$. The remaining generators span the $so(2, 2) \times so(2) = \mathcal{H}'$ subalgebra consisting of

$$\begin{aligned} M_{12} &= i\sigma_{03}/2, \quad M_{13} = \sigma_{31}/2, \quad M_{14} = \sigma_{11}/2, \\ M_{23} &= \sigma_{32}/2, \quad M_{24} = \sigma_{12}/2, \quad M_{34} = -i\sigma_{20}/2, \end{aligned} \quad (34)$$

$M_{\mu\nu} = -M_{\nu\mu}$, and $D = i\sigma_{23}$.

The commutation relations read

$$\begin{aligned}
[D, M_{\mu\nu}] &= 0, \quad [\Gamma_\mu^1, \Gamma_\nu^1] = [\Gamma_\mu^2, \Gamma_\nu^2] = -4M_{\mu\nu}, \quad [\Gamma_\mu^1, \Gamma_\nu^2] = 2D\eta_{\mu\nu}, \\
[D, \Gamma_\mu^1] &= -2\Gamma_\mu^2, \quad [M_{\mu\nu}, \Gamma_\lambda^1] = \eta_{\mu\lambda}\Gamma_\nu^1 - \eta_{\nu\lambda}\Gamma_\mu^1, \\
[D, \Gamma_\mu^2] &= 2\Gamma_\mu^1, \quad [M_{\mu\nu}, \Gamma_\lambda^2] = \eta_{\mu\lambda}\Gamma_\nu^2 - \eta_{\nu\lambda}\Gamma_\mu^2,
\end{aligned} \tag{35}$$

together with the standard commutators for $M_{\mu\nu} \in so(2, 2)$. Also useful are the following anticommutators:

$$\begin{aligned}
\{\Gamma_\mu^1, \Gamma_\nu^2\} &= 4\tilde{M}_{\mu\nu}, \quad \{M_{\mu\nu}, \Gamma_\lambda^1\} = -i\epsilon_{\mu\nu\lambda}{}^\rho \Gamma_\rho^2, \quad \{M_{\mu\nu}, \Gamma_\lambda^2\} = i\epsilon_{\mu\nu\lambda}{}^\rho \Gamma_\rho^1, \\
\{M_{\mu\nu}, M_{\lambda\rho}\} &= -\frac{1}{2}(\eta_{\mu\lambda}\eta_{\nu\rho} - \eta_{\mu\rho}\eta_{\nu\lambda})I - \frac{i}{2}D\epsilon_{\mu\nu\lambda\rho}.
\end{aligned} \tag{36}$$

where $\tilde{M}_{\mu\nu} = i\epsilon_{\mu\nu}{}^{\lambda\tau}M_{\lambda\tau}/2$, $\epsilon_{1234} = 1$.

In the CPW case one deals with the Clifford algebras related to the compact subgroup $SO(4)$:

$$\Gamma_\mu^1 = -\{\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{30}\}, \quad \Gamma_\mu^2 = \{\sigma_{31}, \sigma_{32}, \sigma_{33}, -\sigma_{10}\}, \tag{37}$$

while the remaining generators span the $so(4) \times so(2) = \mathcal{H}$ (maximal compact) subalgebra of $su(2, 2)$:

$$\begin{aligned}
M_{12} &= -i\sigma_{03}/2, \quad M_{13} = i\sigma_{02}/2, \quad M_{14} = i\sigma_{21}/2, \\
M_{23} &= -i\sigma_{01}/2, \quad M_{24} = i\sigma_{22}/2, \quad M_{34} = i\sigma_{23}/2, \quad D = i\sigma_{20}.
\end{aligned} \tag{38}$$

The commutators and anticommutators are the same, but now $\eta_{\mu\nu} = \text{diag}(1, 1, 1, 1)$. In both cases $\lambda = \pm 1$ we have:

$$B = \alpha\Gamma^1 + \beta\Gamma^2 \equiv \alpha^\mu\Gamma_\mu^1 + \beta^\mu\Gamma_\mu^2, \tag{39}$$

with constant $SO(2, 2)$, (resp. $SO(4)$) vectors α, β . Similarly to [9], one can show that

$$\begin{aligned}
B^2 &= (\alpha^2 + \beta^2)I + 4(\alpha \wedge \beta) \cdot \tilde{\mathbf{M}}, \\
B^3 &= \alpha'\Gamma^1 + \beta'\Gamma^2, \\
B^4 &= [(\alpha^2 + \beta^2)^2 + 4(\alpha \wedge \beta)^2]I + 8(\alpha^2 + \beta^2)(\alpha \wedge \beta) \cdot \tilde{\mathbf{M}},
\end{aligned} \tag{40}$$

where $\alpha^2 = \eta_{\mu\nu}\alpha^\mu\alpha^\nu$ etc., and

$$\alpha' = 2\beta \wedge (\beta \wedge \alpha) + (\alpha^2 + \beta^2)\alpha, \quad \beta' = 2\alpha \wedge (\alpha \wedge \beta) + (\alpha^2 + \beta^2)\beta. \tag{41}$$

Leaving the construction of the most general null geodesic solution to a separate publication, here we give the geodesic interpretation of the ‘SWIP’ solutions found recently [11]. They correspond to degenerate B . From (39) one finds

$$\det B = (\alpha^2 - \beta^2)^2 + 4(\alpha\beta)^2. \tag{42}$$

For SAF the standard definition [9] of the ADM mass M , the NUT parameter N , the dilaton and axion charges D, A and electric/magnetic charges Q^n, P^n (assuming $\sigma \rightarrow 1/r$ as $r \rightarrow \infty$) gives

$$\alpha^\mu = (\sqrt{2}P^1, \sqrt{2}P^2, A - N, M + D), \quad \beta^\mu = (-\sqrt{2}Q^1, -\sqrt{2}Q^2, M - D, A + N). \tag{43}$$

The degeneracy condition $\det B = 0$ implies $\alpha^2 = \beta^2$ and $\alpha\beta = 0$, what reduces to

$$D + iA = -\frac{(Q^n + iP^n)^2}{2(M + iN)}. \tag{44}$$

This is a well-known relation for axion–dilaton black holes.

Extremal black holes can be identified with null geodesics. Since

$$dl^2 = \frac{1}{4}\text{Tr}(B^2)(d\sigma)^2, \quad \mathcal{R}_{ij} = \frac{1}{4}\text{Tr}(B^2)\sigma_i\sigma_j, \quad (45)$$

and in this case $\text{Tr}(B^2) = 0$, the three-space is Ricci-flat. According to (40),

$$\text{Tr}(B^2) = 4(\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2), \quad (46)$$

so geodesics are null if $\boldsymbol{\alpha}^2 + \boldsymbol{\beta}^2 = 0$ (what may be fulfilled with non-zero $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in the $SO(2, 2)$ case). For $p = 1$ this condition entails $B^2 = 0$ (*i.e.* collinear $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ [9]), but for $p \geq 2$ it is not necessarily so.

The $p = 2$ TS admits four mutually orthogonal null vectors, and consequently four independent (real) harmonic functions may be incorporated into the geodesic construction [9]. It is convenient to introduce σ -valued vectors \mathbf{a} and \mathbf{b} as linear combinations $a^\mu = \alpha_{(A)}^\mu \sigma_{(A)}$ and $b^\mu = \beta_{(A)}^\mu \sigma_{(A)}$, $A = 1, \dots, 4$, so that $B = \mathbf{a}\boldsymbol{\Gamma}^1 + \mathbf{b}\boldsymbol{\Gamma}^2$ (only four components of σ -valued vectors are independent in view of the consistency conditions [9]), then

$$\mathcal{M} = A \left(I_4 + \mathbf{a}\boldsymbol{\Gamma}^1 + \mathbf{b}\boldsymbol{\Gamma}^2 + 2(\mathbf{a} \wedge \mathbf{b}) \cdot \tilde{\mathbf{M}} \right). \quad (47)$$

Comparing with (13) one finds

$$\begin{aligned} f^{-1} &= (1 + a^4)(1 + b^3) - a^3b^4, \quad e^{2\phi} = f \left[(1 + a^4)^2 + (b^4)^2 \right], \\ v^n &= -f \left[(1 + a^4)b^n - b^4a^n \right], \quad u^n = f \left[(1 + b^3)a^n - a^3b^n \right], \\ \kappa &= \left[(1 + a^4)a^3 + (1 + b^3)b^4 \right] / \left[(1 + a^4)^2 + (b^4)^2 \right], \quad \chi = f \left(a^3 - b^4 \right). \end{aligned} \quad (48)$$

Actually eight components of \mathbf{a}, \mathbf{b} depend on four real harmonic functions, say, a^3, a^4, b^3, b^4 , from which one can form two complex harmonic functions

$$\mathcal{H}_1 = a^3 + i(1 + b^3), \quad \mathcal{H}_2 = (1 + a^4) + ib^4. \quad (49)$$

Then

$$f^{-1} = \text{Im}(\mathcal{H}_1 \bar{\mathcal{H}}_2), \quad z = \frac{\mathcal{H}_1}{\mathcal{H}_2}, \quad \chi = f \left(\text{Re}\mathcal{H}_1 - \text{Im}\mathcal{H}_2 \right), \quad (50)$$

what gives the metric and axidilaton part of ‘SWIP’ [11]. For vector fields a different gauge was used in [11], namely $v_\infty^n - iu_\infty^n = k^n$, where $k^n = k'^n + ik''^n$ is a complex constant vector satisfying conditions $(k^n)^2 = 0$, $|k^n|^2 = 2$. In our formalism this correspond to the following choice of the remaining components of \mathbf{a}, \mathbf{b} (consistent with $\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{a}\mathbf{b} = 0$):

$$\begin{aligned} a^1 &= k'^1 a^3 + k''^1 a^4, \quad a^2 = k'^2 a^3 + k''^2 a^4, \\ b^1 &= k'^1 b^3 + k''^1 b^4, \quad b^2 = k'^2 b^3 + k''^2 b^4, \end{aligned} \quad (51)$$

accompanied by shifts $v'^n = v^n + k'^n, u'^n = u^n - k''^n$. The twist potential then undergoes a transformation which makes it zero, while the rest of the solution will read

$$\begin{aligned} v'^n &= f \text{Re}(k^n \mathcal{H}_2), \quad u'^n = f \text{Re}(k^n \mathcal{H}_1), \\ h_{ij} &= \delta_{ij}, \quad \epsilon^{ijk} \partial_j \omega_k = \text{Re} \left[\mathcal{H}_1 \partial_i \bar{\mathcal{H}}_2 - \bar{\mathcal{H}}_2 \partial_i \mathcal{H}_1 \right]. \end{aligned} \quad (52)$$

The isotropy condition $\text{Tr}B^2 = 0$ in terms of charges is equivalent to the force balance [9, 10]:

$$M^2 + N^2 + D^2 + A^2 = (Q^n)^2 + (P^n)^2. \quad (53)$$

As it was noted, for $p \geq 2$ null geodesic solutions with $\det B = 0$ form two subclasses: those with collinear and those with non-collinear α, β . In the first case $B^2 = 0$, hence the second condition arises:

$$M^2 + N^2 = D^2 + A^2. \quad (54)$$

The conditions (53-54) together are equivalent to both BPS bounds of the $N = 4$ theory saturated, what corresponds to the $N = 2$ residual SUSY [11]. For non-collinear α, β only the force balance condition holds (the $N = 1$ residual SUSY).

Our second example is the CPW counterpart of the DARN-NUT solution [10]. It is well-known that certain CPW in the collision region map onto black hole interiors [12]. Like black holes, CPW belong to spacetimes with two commuting Killing vectors, so one can further specialize three-dimensional coordinates as follows

$$h_{ij}dx^i dx^j = e^{2\gamma} (d\rho^2 - dz^2) - \rho^2 dx^2 \quad (55)$$

(the second Killing vector is ∂_x , and $\omega_i = \omega \delta_{ix}$ in (3)). Consider degenerate B , putting without loss of generality $\alpha^2 = \beta^2 = 1$, $\alpha\beta = 0$ with non-collinear α, β . Then

$$A\mathcal{M} = I_4 \cosh^2 \sigma + 2(\alpha \wedge \beta) \cdot \tilde{\mathbf{M}} \sinh^2 \sigma + \frac{1}{2} (\alpha \cdot \mathbf{\Gamma}^1 + \beta \cdot \mathbf{\Gamma}^2) \sinh 2\sigma. \quad (56)$$

Note that the three-space in the CPW case can not be Ricci-flat (for the $SO(4)$ metric $\alpha^2 + \beta^2 = 0$ implies $\alpha = \beta = 0$), with our normalization

$$\mathcal{R}_{ij} = 2\sigma_i \sigma_j. \quad (57)$$

Appropriate harmonic functions should be found together with γ in a self-consistent way. A simple solution is

$$\sigma = \frac{1}{2} \ln \left(\frac{1+\tau}{1-\tau} \right), \quad e^{2\gamma} = \frac{1-\tau^2}{\tau^2 - \zeta^2}, \quad (58)$$

where new coordinates correspond to

$$\rho^2 = (1 - \tau^2)(1 - \zeta^2), \quad z = -\tau\zeta. \quad (59)$$

This results in the following family of $N = 4$ CPW:

$$ds^2 = \Sigma \left(\frac{d\tau^2}{1-\tau^2} - \frac{d\zeta^2}{1-\zeta^2} \right) - \frac{1-\tau^2}{\Sigma} (dy - Q\zeta dx)^2 - (1-\zeta^2)\Sigma dx^2, \quad (60)$$

where $\Sigma = 1 + (\beta^3 - \alpha^4)\tau + (\alpha \wedge \beta)^{34}\tau^2$, $Q = \beta^4 + \alpha^3$ (with $e^{\tau\zeta x} = 1$), and material fields are

$$-v^n = \tau [\beta^n + \tau(\alpha \wedge \beta)^{n4}] / \Sigma, \quad -u^n = \tau [\alpha^n + \tau(\alpha \wedge \beta)^{n3}] / \Sigma, \\ e^{-2\phi} = \Sigma / \Delta, \quad \kappa = \tau [\beta^4 - \alpha^3 + (\alpha^3 \alpha^4 + \beta^3 \beta^4)\tau] / \Delta, \quad (61)$$

with $\Delta = 1 - 2\alpha^4\tau + [(\alpha^4)^2 + (\beta^4)^2]\tau^2$. For $\alpha^1 = \beta^1 = \alpha^2 = \beta^2 = 0$ this solution may be interpreted as Ferrari-Ibanez-Bruni CPW [13] (with $(\alpha \wedge \beta)^{34} = 1$), or as collinear impulsive CPW with dilaton and axion $((\alpha \wedge \beta)^{34} = -1, Q = 0)$. General solution (60,61) may be considered as the CPW counterpart of the DARN-NUT solution [10]. Indeed, if one put in the latter $r = M_0(t+1) - r_0^-$, $\cos \theta = \zeta$, $\varphi = x$, $t = M_0 y$ (notation of [10]), the collision region of (60,61) is recovered with $N = -Q/(2M_0)$. Note that the extremality (BPS) limit of the DARN-NUT solution corresponds to $(\alpha \wedge \beta)^{34} = 0$, the relevant CPW then has Σ linear in τ , but the above coordinate map becomes singular.

A notable feature of three-dimensional sigma-models on symmetric spaces is that their further two-dimensional reduction generates (modified) chiral equations which belong to the class of integrable systems (for a simple derivation see, *e.g.* [4]). Both vacuum Einstein's and $p = 1$ EMDA theory are known to admit two alternative Lax pairs related by the Kramer–Neugebauer (KN) map [14]. Here we generalize this construction to arbitrary p . Let indices $A, B = 1, 2$ correspond to coordinates on the two-surface orthogonal to Killing orbits. Define

$$h_{AB} = e^{2\gamma} G_{AB}, \quad h_{xx} = \lambda \rho^2, \quad G_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad \epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (62)$$

Introduce instead of u^n the non-dualized potentials a^n via $F_{Ax}^n = \nabla_A a^n / \sqrt{2}$, and let $q^n = a^n + \omega v^n$, $b = B_{yx}$, (a component of the Kalb–Ramond field underlying the Peccei–Quinn axion κ). Then the ‘Matzner–Misner’ counterpart of the ‘potential’ matrix \mathcal{Q} for $p = 2$ will be the following hermitean matrix

$$\Omega = \begin{pmatrix} \omega & -(q^1 - iq^2) \\ -(q^1 + iq^2) & q^n v^n - b \end{pmatrix} \quad (63)$$

(for $p = 1$ a similar representation was given in [14]). Its arbitrary- p generalization is straightforward:

$$\Omega = w^0 I_k + w^a \gamma_a, \quad 2w^0 = \omega - b + q^n v^n, \quad w^n = -q^n, \quad 2w^{p+1} = \omega + b - q^n v^n. \quad (64)$$

From the equations of motion one can derive the following relation between Ω and \mathcal{Q} :

$$\nabla Q = -\rho^{-1} P \tilde{\nabla} \Omega P, \quad (65)$$

where $\nabla_A = (\partial_1, \partial_2)$, $\tilde{\nabla}^A = \epsilon^{AB} \nabla_B$, and A, B are raised and lowered by G_{AB} . A ‘Matzner–Misner’ matrix can now be introduced

$$\mathcal{F} = -\rho^{-1} \begin{pmatrix} P & -P \Omega \\ -\Omega P & \Omega P \Omega - \lambda \rho^2 P^{-1} \end{pmatrix}, \quad (66)$$

which satisfies chiral equations of the same type as \mathcal{M} :

$$\nabla^A (\rho \mathcal{F}^{-1} \nabla_A \mathcal{F}) = 0, \quad \nabla^A (\rho \mathcal{M}^{-1} \nabla_A \mathcal{M}) = 0. \quad (67)$$

Variables entering \mathcal{F} are related nonlocally to the sigma-model variables in \mathcal{M} . Now, by definition, a KN map is a *local* relation between two alternative forms of chiral equations. Comparing (13) and (65) one finds that the map

$$\mathcal{Q} \rightarrow \sqrt{-\lambda} \Omega, \quad \mathcal{P} \rightarrow \rho \mathcal{P}^{-1}, \quad (68)$$

transform the equations for $(\mathcal{P}, \mathcal{Q})$ into those for (\mathcal{P}, Ω) . This opens a way of further development as discussed in [14].

Hence the Ernst-type picture of the $N = 4$ supergravity amounts to the *pseudounitary* embedding of the three-dimensional U-duality group. Previously found symplectic representation of the EMDA theory is valid uniquely for the one-vector truncation. Meanwhile its basic features such as an existence in the two-dimensional case of the Matzner–Misner counterpart and the Kramer–Neugebauer mapping remain valid thus opening the way to application of various techniques of the theory of integrable systems.

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